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THE SIGNIFICANCE OF (AN)ISOTROPIC VISCOELASTIC POISSON RATIO STRESS AND TIME DEPENDENCIES†

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Abstract—Although viscoelastic moduli may be linear, Poisson ratios (PR) are always nonlinear functions of pairs of normal strains. This is equally true for physical PR defined in the real time space and for pseudo PR differently derived in the Fourier transform (FT) domain. It is shown analytically that only if anisotropic or isotropic viscoelastic moduli and relaxation or creep functions are characterized by identical time functions in all directions and stresses are constants or at most temporal and spatial separable functions, then corresponding PR must be time independent. Under all other conditions PR are proven to be time, stress and thermal expansion dependent through time integrals, although physical and pseudo PR are shown to be functionally unrelated. The consequences of PR nonlinearities are that their uniaxially determined values are not applicable to other uniaxial loadings with different time histories or to multiaxial loadings and thermal expansions, if the latter are present. Similarly, isotropic PR cannot generally be determined solely from viscoelastic Young's and shear moduli, even for linear materials. Consequently, viscoelastic material property characterization in terms of PR is not unique and viscoelastic responses are best described in terms of creep or relaxation functions. Anisotropic and isotropic viscoelastic PR time effects are investigated analytically and evaluated numerically. © 1998 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

The phenomenon of deformations normal to the loading axis resulting from 1-D forces, known as the Poisson ratio effect (Poisson 1829a, b), has received extensive attention in the theories of elasticity and plasticity. Historically, being deeply rooted in the usage of elastic and plastic metals, Poisson ratios have enjoyed the advantage of relatively easy experimental determinations. Electric strain gages can be used to great advantage on metals, because their resistance wires have the same elastic moduli as steel and aluminum. However, when applied to "softer" (less "rigid") materials, such as polymers, rubbery materials or composites, strain gage moduli are five to six orders of magnitude higher than those of the tested materials. Indeed, here these gages are a detriment to such experiments since they serve as reinforcements to the actual material. Under these conditions such strain readings

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are the values specific to gage wire displacements and do not properly reflect true actual strain magnitudes in the specimen.

By experimental necessity, multiaxial polymer material property determinations are conducted on thin strips or high aspect ratio thin plates under uniaxial loadings and multiaxial stress fields. While accurate measurements of strains and/or deformations in the directions of 1-D loads are generally achievable, because of the specimen's relatively large dimensions in the loaded direction, this facility is lost in the other orientations because of their relatively small deflections. These inherent problems pose limitations and make accurate experimental measurements of Poisson ratios, moduli and of the associated multiaxial strains extremely difficult when under the simplest service conditions and nigh impossible during manufacture (cure) when material phase shifts and high temperature gradients are ever present.

These difficulties are further compounded in viscoelastic materials by the fact that all moduli and Poisson ratios are time dependent material properties (Gurtin and Sternberg, 1963; Hilton, 1964a; Hilton and Dong, 1964b) and for anisotropic viscoelasticity they additionally take on distinct tie functions in different directions. As a consequence of the operational restrictions associated with strain measurements along and particularly normal to the 1-D loading axis, many experimental researchers have assumed identical time functions for all anisotropic viscoelastic moduli, thus, as will be shown subsequently, reducing the attendant Poisson ratios to constants. Such assumptions severely cloud the proper characterization of real viscoelastic materials and leave analysis in the dark regarding the true behavior of anisotropic moduli or compliances and relaxation and/or creep functions.

For a comprehensive and rigorous treatment of anisotropic elasticity, which forms the background for anisotropic viscoelasticity, the reader is referred to Ting (1996).

2. ANALYSIS

As an introduction to this topic consider a general linear anisotropic viscoelastic material in a Cartesian coordinate system $x = x_k$ with $k = 1, 2, 3$ where, under proper conditions, the Fourier transforms (FT) of the constitutive relations may be written in the form (Hilton and Dong, 1964a, b)

$$\bar{\bar{\sigma}}_{kl}(x,\omega) = \bar{\bar{E}}_{klmn}(x,\omega)\bar{\bar{\varepsilon}}_{mn}(x,\omega) - \bar{\bar{Q}}_{kl}(x,\omega)\overline{\overline{\mathscr{A}\mathscr{T}}}(x,\omega) \tag{2.1a}$$

or alternatively

$$\bar{\bar{\varepsilon}}_{kl}(x,\omega) = \bar{\bar{C}}_{klmn}(x,\omega)\bar{\bar{\sigma}}_{mn}(x,\omega) + \bar{\bar{q}}_{kl}(x,\omega)\overline{\overline{\mathscr{A}\mathscr{T}}}(x,\omega) \tag{2.1b}$$

where $\bar{\bar{E}}_{klmn}$ and $\bar{\bar{C}}_{klmn}$ are, respectively, complex moduli and compliances with interdependences $\bar{\bar{E}}_{klmn} = 1/\bar{\bar{C}}_{klmn}$, q_{kl} are thermal expansion functions, $\mathscr{A}$ is the coefficient of thermal expansion, $\mathscr{T}$ the temperature w.r.t. a reference temperature T_0 where the strains are zero and with the total temperature defined by $T = T_0 + \mathscr{T}$. Repeated indices, unless underlined, indicate summations. The overscores indicate FT of time functions, such that

$$\bar{\bar{F}}(x,\omega) = \int_{-\infty}^{\infty} F(x,t)\exp(-i\omega t)\,\mathrm{d}t \quad \text{or}\ \bar{\bar{F}}(x,\omega) = \int_{-\infty}^{\infty} \hat{F}(x,\xi)\exp(-i\omega\xi)\,\mathrm{d}\xi$$

where $i = \sqrt{-1}$. The FT are applicable only if $T = T(x)$ or if one assumes that all the anisotropic shift functions a_{klmn} and reduced times ξ_{klmn} are, respectively, equal to themselves, such that they can be represented by a single ξ

$$\xi(x,t) \equiv \xi_{klmn}(x,t) = \int_0^t a_{klmn}[x, t', T(x,t'), M(x,t'), \alpha(x,t',T)]\,\mathrm{d}t'$$

and

$$\xi(x,t) \equiv \xi_{klmnT}(x,t) = \int_0^t a_{klmnT}[x, t', T(x,t'), M(x,t'), \alpha(x,t',T)]\,\mathrm{d}t'$$

where M is the moisture content and with time-temperature shift functions $a \equiv a_{klmn} \equiv a_{klmnT}$. During manufacture the degree of cure α has values ranging from $0 \leqslant \alpha \leqslant 1$, while in service $\alpha = 1$ (Yi *et al.* 1995, 1996, 1997).

For the purpose of an initial simplified analysis and discussion of viscoelastic time effects of Poisson ratios, let the state of stress be limited to a uniaxial one with isothermal orthotropic properties, such that $\sigma_{11} \neq 0$ while all other $\sigma_{kl} = 0$ and $\mathscr{T} = 0$, resulting in a reduced set of relations (2.1) given by

$$\bar{\bar{\sigma}}_{11} = \bar{\bar{E}}_{11}\bar{\bar{\varepsilon}}_{11} + \bar{\bar{E}}_{12}\bar{\bar{\varepsilon}}_{22} + \bar{\bar{E}}_{13}\bar{\bar{\varepsilon}}_{33} \quad \text{or}\,\bar{\bar{\varepsilon}}_{11} = \bar{\bar{C}}_{11}\bar{\bar{\sigma}}_{11} \tag{2.2a, b}$$

$$0 = \bar{\bar{E}}_{12}\bar{\bar{\varepsilon}}_{11} + \bar{\bar{E}}_{22}\bar{\bar{\varepsilon}}_{22} + \bar{\bar{E}}_{23}\bar{\bar{\varepsilon}}_{33} \quad \text{or}\,\bar{\bar{\varepsilon}}_{22} = \bar{\bar{C}}_{12}\bar{\bar{\delta}}_{11} \tag{2.3a, b}$$

$$0 = \bar{\bar{E}}_{13}\bar{\bar{\varepsilon}}_{11} + \bar{\bar{E}}_{23}\bar{\bar{\varepsilon}}_{22} + \bar{\bar{E}}_{33}\bar{\bar{\varepsilon}}_{33} \quad \text{or}\,\bar{\bar{\varepsilon}}_{33} = \bar{\bar{C}}_{13}\bar{\bar{\sigma}}_{11} \tag{2.4a, b}$$

with the usual symmetry conditions on all functions in the form $F_{kl} = F_{lk}$ for $k \neq l$ and where

$$E_{\underline{kk}} \neq E_{\underline{ll}} \neq E_{lk} \quad \text{for}\,k \neq l \tag{2.5}$$

For any general loading and/or thermal state, viscoelastic PR are definable in the real time domain in the same physical sense as the original elastic ones (Poisson 1829a, b), i.e. the ratio of normal strain pairs in mutually perpendicular directions, to whit

$$\nu_{kl}(x,t) = -\frac{\varepsilon_{\underline{ll}}(x,t)}{\varepsilon_{\underline{kk}}(x,t)} \quad \text{for}\,k \neq l \tag{2.6}$$

with no summation over the underlined indices and with the understanding that the above physical definition of ν_{kl} mandates that

$$\bar{\bar{\nu}}_{kl}(x,\omega) \neq -\frac{\bar{\bar{\varepsilon}}_{\underline{ll}}(x,\omega)}{\bar{\bar{\varepsilon}}_{\underline{kk}}(x,\omega)} \quad \text{and} \quad \nu_{kl} \neq \nu_{lk} \quad \text{for}\,k \neq l \tag{2.7}$$

Therefore, the elastic/viscoelastic analogy does not apply to Poisson ratios including those for isotropic materials, even though it holds individually for strain, stress and displacement tensors. This is due to the fact that ν_{kl} in eqns (2.6) are related nonlinearly to pairs of normal strain components, including cases when materials obey linear constitutive relations.

Alternately, pseudo Poisson ratios can be defined as shown in Appendix A in the FT ω space based on the equality of eqns (2.7) and inequality of eqn (2.6) and shown in Table 2.

Substituting general viscoelastic constitutive relations into (2.6), yields

Table 1. Some properties of viscoelastic Poisson ratios ν_{kl}

Case	Conditions	Stresses	Temperature $\mathscr{T}$	Moduli	$\nu_{kl}(x, t)$	Eqn No.
1.1	Isotropic	Uniaxial const.	0	STF	$\nu_{kl}(x)$	(2.10)
1.2	Isotropic	Uniaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu_{kl}(x)$	(2.10)
1.3	Isotropic	Uniaxial $\sigma_{kl}(x, t)$	0	STF	$\nu_{kl}(x)$	(2.10)
1.4	Isotropic	Uniaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(2.9)
1.5	Isotropic	Multiaxial const.	0	STF	$\nu_{kl}(x)$	(2.9)
1.6	Isotropic	Multiaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu_{kl}(x, \sigma)$	(2.9)
1.7	Isotropic	Multiaxial $\sigma_{kl}(x, t)$	0	STF	$\nu_{kl}(x, t, \sigma)$	(2.9)
1.8	Isotropic	Multiaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(2.9)
1.9	Anisotropic	Uniaxial const.	0	STF	$\nu_{kl}(x)$	(2.8)
1.10	Anisotropic	Uniaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu_{kl}(x)$	(2.8)
1.11	Anisotropic	Uniaxial $\delta_{kl}(x, t)$	0	STF	$\nu_{kl}(x, t, \sigma)$	(2.8)
1.12	Anisotropic	Uniaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(2.8)
1.13	Anisotropic	Uniaxial const.	0	DTF	$\nu_{kl}(x, t)$	(2.8)
1.14	Anisotropic	Uniaxial $g(t)\sigma^*_{kl}(x)$	0	DTF	$\nu_{kl}(x, t, \sigma)$	(2.8)
1.15	Anisotropic	Uniaxial $\sigma_{kl}(x, t)$	0	DTF	$\nu_{kl}(x, t, \sigma)$	(2.8)
1.16	Anisotropic	Uniaxial $\sigma_{kl}(x, t)$	$\neq 0$	DTF	$\nu_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(2.8)
1.17	Anisotropic	Multiaxial const.	0	STF	$\nu_{kl}(x)$	(2.8)
1.18	Anisotropic	Multiaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu_{kl}(x, \sigma)$	(2.8)
1.19	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	0	STF	$\nu_{kl}(x, t, \sigma)$	(2.8)
1.20	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(2.8)
1.21	Anisotropic	Multiaxial const.	0	DTF	$\nu_{kl}(x, t)$	(2.8)
1.22	Anisotropic	Multiaxial $g(t)\sigma^*_{kl}(x)$	0	DTF	$\nu_{kl}(x, t, \sigma)$	(2.8)
1.23	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	0	DTF	$\nu_{kl}(x, t, \sigma)$	(2.8)
1.24	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	$\neq 0$	DTF	$\nu_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(2.8)

STF = same time functions. DTF = distinct time functions in all directions.

Table 2. Some properties of viscoelastic pseudo Poisson ratios ν^{a}_{kl}

Case	Conditions	Stresses	Temperature $\mathscr{T}$	Moduli	$\nu^{a}_{kl}(x, t)$	Eqn No.
2.1	Isotropic	Uniaxial const.	0	STF	$\nu^{a}_{kl}(x)$	(A.3)
2.2	Isotropic	Uniaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu^{a}_{kl}(x)$	(A.3)
2.3	Isotropic	Uniaxial $\sigma_{kl}(x, t)$	0	STF	$\nu^{a}_{kl}(x)$	(A.3)
2.4	Isotropic	Uniaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu^{a}_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(A.6)
2.5	Isotropic	Multiaxial const.	0	STF	$\nu^{a}_{kl}(x)$	(A.6)
2.6	Isotropic	Multiaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu^{a}_{kl}(x, \sigma)$	(A.6)
2.7	Isotropic	Multiaxial $\sigma_{kl}(x, t)$	0	STF	$\nu^{a}_{kl}(x, \sigma)$	(A.6)
2.8	Isotropic	Multiaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu^{a}_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(A.6)
2.9	Anisotropic	Uniaxial const.	0	STF	$\nu^{a}_{kl}(x)$	(A.6)
2.10	Anisotropic	Uniaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu^{a}_{kl}(x, \sigma)$	(A.6)
2.11	Anisotropic	Uniaxial $\sigma_{kl}(x, t)$	0	STF	$\nu^{a}_{kl}(x, t, \sigma)$	(A.6)
2.12	Anisotropic	Uniaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu^{a}_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(A.6)
2.13	Anisotropic	Uniaxial const.	0	DTF	$\nu^{a}_{kl}(x, t)$	(A.6)
2.14	Anisotropic	Uniaxial $g(t)^*_{kl}(x)$	0	DTF	$\nu^{a}_{kl}(x, t)$	(A.6)
2.15	Anisotropic	Uniaxial $\sigma_{kl}(x, t)$	0	DTF	$\nu^{a}_{kl}(x, t)$	(A.6)
2.16	Anisotropic	Uniaxial $\sigma_{kl}(x, t)$	$\neq 0$	DTF	$\nu^{a}_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(A.6)
2.17	Anisotropic	Multiaxial const.	0	STF	$\nu^{a}_{kl}(x, t)$	(A.6)
2.18	Anisotropic	Multiaxial $g(t)\sigma^*_{kl}(x)$	0	STF	$\nu^{a}_{kl}(x, \sigma)$	(A.6)
2.19	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	0	STF	$\nu^{a}_{kl}(x, t, \sigma)$	(A.6)
2.20	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	$\neq 0$	STF	$\nu^{a}_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(A.6)
2.21	Anisotropic	Multiaxial const.	0	DTF	$\nu^{a}_{kl}(x, t)$	(A.6)
2.22	Anisotropic	Multiaxial $g(t)\sigma^*_{kl}(x)$	0	DTF	$\nu^{a}_{kl}(x, t, \sigma)$	(A.6)
2.23	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	0	DTF	$\nu^{a}_{kl}(x, t, \sigma)$	(A.6)
2.24	Anisotropic	Multiaxial $\sigma_{kl}(x, t)$	$\neq 0$	DTF	$\nu^{a}_{kl}(x, t, \sigma, \mathscr{A}\mathscr{T})$	(A.6)

STF = same time functions. DTF = distinct time functions in all directions.

$$v_{kl}(x,t) = -\frac{\displaystyle\int_0^t C_{\underline{ll}mn}(x,t,t')\sigma_{mn}(x,t')\,\mathrm{d}t' + \int_0^t q_{\underline{ll}}(x,t,t')\mathscr{A}\mathscr{T}(x,t')\,\mathrm{d}t'}{\displaystyle\int_0^t C_{\underline{kk}op}(x,t,t')\sigma_{op}(x,t')\,\mathrm{d}t' + \int_0^t q_{\underline{kk}}(x,t,t')\mathscr{A}\mathscr{T}(x,t')\,\mathrm{d}t'} \quad \text{for } k \neq l \tag{2.8}$$

Each viscoelastic PR, then, is not only dependent on the compliances (or moduli) but also on all nonvanishing stress tensor components (including shears) and temperature histories. A similar phenomenon has been noted by Hilton and Peichocki (1962) for viscoelastic shear centers.

In the isotropic case, eqns (2.8) reduce to

$$v_{kl}(x,t) = -\frac{\displaystyle\sum_{m=1}^{3}\left[\int_0^t C_{\underline{ll}mm}(x,t,t')\sigma_{mm}(x,t')\,\mathrm{d}t'\right] + \int_0^t q(x,t,t')\mathscr{A}\mathscr{T}(x,t')\,\mathrm{d}t'}{\displaystyle\sum_{m=1}^{3}\left[\int_0^t C_{\underline{kk}mm}(x,t,t')\sigma_{mm}(x,t')\,\mathrm{d}t'\right] + \int_0^t q(x,t,t')\mathscr{A}\mathscr{T}(x,t')\,\mathrm{d}t'} \quad \text{for } k \neq l \tag{2.9}$$

making v_{kl} still both stress and thermal expansion time history dependent. The same holds true for the simplest isothermal uniaxial loading conditions represented by eqns (2.2) to (2.4), which yield

$$v_{kl}(x,t) = -\frac{\displaystyle\int_0^t C_{\underline{ll}11}(x,t,t')\sigma_{11}(x,t')\,\mathrm{d}t'}{\displaystyle\int_0^t C_{\underline{kk}11}(x,t,t')\sigma_{11}(x,t')\,\mathrm{d}t'} \quad k \neq l \tag{2.10}$$

and preserve the single stress tensor component loading history dependence of v.

Isotropic viscoelastic materials can also be characterized in terms of shear (G) and bulk (K) moduli and when FT are applicable, the following relations exist

$$\begin{aligned}
\bar{\bar{C}}_{1111} &= \bar{\bar{C}}_{2222} = \bar{\bar{C}}_{3333} = \frac{1}{\bar{\bar{E}}} = \frac{1+\bar{\bar{G}}/\bar{\bar{K}}}{3\bar{\bar{G}}} \\
\bar{\bar{C}}_{1212} &= \bar{\bar{C}}_{2323} = \bar{\bar{C}}_{1313} = \frac{1}{\bar{\bar{G}}} \\
\bar{\bar{C}}_{12\underline{mm}} &= \bar{\bar{C}}_{13\underline{mm}} = \bar{\bar{C}}_{23\underline{mm}} = \frac{1}{6\bar{\bar{G}}} - \frac{1}{3\bar{\bar{K}}}
\end{aligned} \tag{2.11}$$

(See Appendix A for some applications of eqns (2.11).) The effects of stresses and thermal expansions on expressions (2.8) to (2.10) are shown in Table 1 and discussed in the next section. Poisson ratios for nonlinear viscoelastic materials remain prescribed by eqns (2.6), but their FT cannot be defined, and the above analyses and observations are inapplicable.

Anisotropic and isotropic viscoelastic moduli are expressible in Prony series in the time domain, such that for linear materials

$$\begin{aligned}
E_{klmn}(x,t) &= \frac{\partial \phi_{klmn}(x,t)}{\partial t} \\
&= \sum_{p=0}^{N^{\underline{klmn}}} \tilde{\phi}_{\underline{klmnp}}(x) \exp\left\{-\int_0^t \frac{\mathrm{d}t'}{\tau_{\underline{klmnp}}[x, T(x,t'), M(x,t'), \alpha(x,t',T)]}\right\}
\end{aligned} \tag{2.12}$$

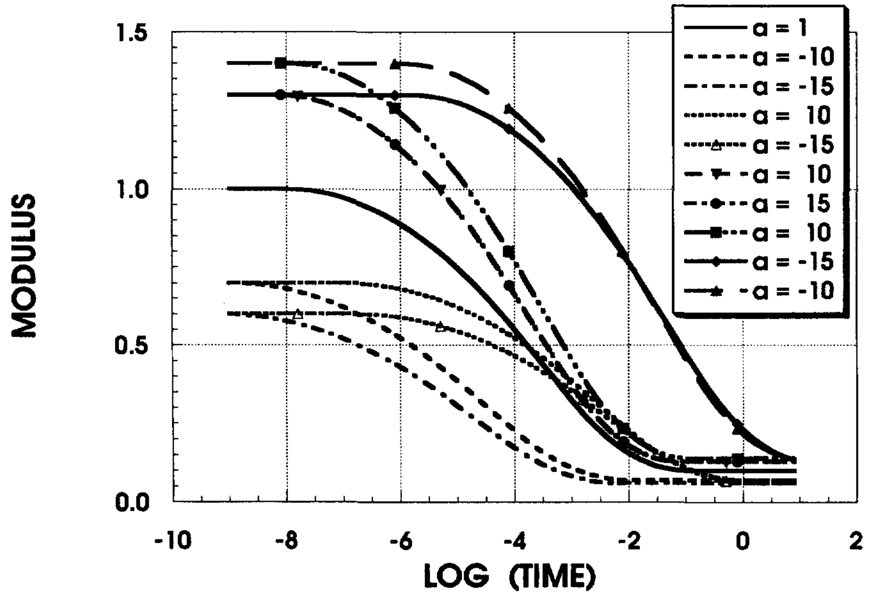

Fig. 1. Effects of $E(0)$ and time shifts on moduli.

where $\phi_{klmn}(x,t)$ are anisotropic nonhomogeneous relaxation functions. The coefficients $\tilde{\phi}_{klmnp}$, relaxation times $\tau_{klmnp}(x,t)$ and the number of series terms N^{klmn} are all distinct directional material properties. The coefficients $\tilde{\phi}_{klmnp}$ are measures of the magnitude of the moduli, while the relaxation times determine the relative shifts on the time scale (Fig. 1). In particular, certain predetermined values are of importance, such as

$$\tau_{klmn0} = \infty \qquad \tilde{\phi}_{klmn0}(x) = \phi_{klmn}(x,\infty) = \phi^{\infty}_{klmn}(x)$$

$$\phi^{0}_{klmn}(x) = \phi_{klmn}(x,0) = \sum_{p=0}^{N^{\underline{klmn}}} \tilde{\phi}_{\underline{klmnp}}(x) \tag{2.13}$$

The simplest anisotropic model, which does not possess these restrictions, is one where amplitude and time shift multipliers are introduced. This characterization encompasses identical N, $\tilde{\phi}_p$ and τ_p values in all directions, while being subjected to directionally sensitive distinct amplitudes and time dependences through orientationally influenced shift parameters a_{klmn}, and is represented by

$$E_{klmn}(x,t) = A_{\underline{klmn}} \sum_{p=1}^{N} \tilde{\phi}_p(x) \exp\left\{-a_{\underline{klmn}} \int_0^t \frac{\mathrm{d}t'}{\tau_p[x, T(x,t'), M(x,t'), \alpha(x,t',T)]}\right\} \tag{2.14}$$

with $\phi^0_{klmn} = A_{klmn}\phi^0$ and $\phi^{\infty}_{klmn} = A_{klmn}\phi^{\infty}$ as seen in Fig. 1.

Similarly, elastic Poisson ratios can be derived from elastic constitutive relations equivalent to eqn (2.1) to yield

$$\nu^e_{kl}(x,t) = -\frac{\varepsilon^e_{\underline{ll}}(x,t)}{\varepsilon^e_{\underline{kk}}(x,t)} = -\frac{C^e_{\underline{ll}mn}(x)\sigma^e_{mn}(x,t) + \mathscr{A}_{\underline{ll}}\mathscr{T}(x,t)}{C^e_{\underline{kk}mn}(x)\sigma^e_{mn}(x,t) + \mathscr{A}_{\underline{kk}}\mathscr{T}(x,t)} \tag{2.15}$$

Table 3 shows the relation of the stresses and thermal expansions to the elastic ν^e_{kl}.

Table 3. Some properties of elastic poisson ratios v^e_{kl}

Case	Conditions	Stresses	Temperature $\mathscr{T}$	$v^e_{kl}(x,t)$	Eqn No.
3.1	Isotropic	Uniaxial $\sigma_{kl}(x,t)$	0	$v^e_{kl}(x)$	(2.15)
3.2	Isotropic	Uniaxial $\sigma_{kl}(x,t)$	$\neq 0$	$v^e_{kl}(x,\sigma,\mathscr{A}\mathscr{T})$	(2.15)
3.3	Isotropic	Multiaxial $\sigma_{kl}(x,t)$	0	$v^e_{kl}(x,\sigma)$	(2.15)
3.4	Isotropic	Multiaxial $\sigma_{kl}(x,t)$	$\neq 0$	$v^e_{kl}(x,\sigma,\mathscr{A}\mathscr{T})$	(2.15)
3.5	Anisotropic	Uniaxial $\sigma_{kl}(x,t)$	0	$v^e_{kl}(x)$	(2.15)
3.6	Anisotropic	Uniaxial $\sigma_{kl}(x,t)$	$\neq 0$	$v^e_{kl}(x,\sigma,\mathscr{A}\mathscr{T})$	(2.15)
3.7	Anisotropic	Multiaxial $\sigma_{kl}(x,t)$	0	$v^e_{kl}(x,\sigma)$	(2.15)
3.8	Anisotropic	Multiaxial $\sigma_{kl}(x,t)$	$\neq 0$	$v^e_{kl}(x,\sigma,\mathscr{A}\mathscr{T})$	(2.15)

3. DISCUSSION

The proper definition of viscoelastic Poisson ratios is the clue to the discussion that follows. Aside from what has been called in this paper the physical (v_{kl}, eqns (2.6)) and the pseudo ($v^{\mathbf{a}}_{kl}$, eqns (A1)) PR, there remains the issue of loading and thermal expansions time histories to be settled. Additionally, the question of whether either the proper or the pseudo definition is limited to uniaxial stresses at isothermal conditions needs to be explored. If the former is true, then as will be shown subsequently, strain data from multiaxial stress experiments cannot be used for uniaxial load Poisson ratio determinations because the viscoelastic v_{kl} and $v^{\mathbf{a}}_{kl}$ are load history dependent. Or conversely, uniaxially determined viscoelastic v_{kl} and $v^{\mathbf{a}}_{kl}$ are not universal and exportable to other multiaxial load histories. Linear compliances, moduli and creep and relaxation functions are stress independent and, hence, load history insensitive, but PR by their very nature are nonlinear functions of pairs of perpendicular tensile or compressive strains. Even for uniaxial stress fields, additional complications arise in the experimental determination and definition of both isotropic and anisotropic v_{kl} and $v^{\mathbf{a}}_{kl}$ if thermal expansions are present.

It can be seen from eqns (2.6) for the physical Poisson ratios v_{kl} and (A1) for the pseudo $v^{\mathbf{a}}_{kl}$ that these two definitions are incompatible with each other. Consider the simplest possible set of conditions for an isothermal uniaxial time independent loading with compliances obeying identical time functions. Then from eqns (2.8) and (A6), one obtains

$$v_{kl}(x,t) = v^e_{kl}(x) = -\frac{C^0_{\underline{ll}11}(x)}{C^0_{\underline{kk}11}(x)} \tag{3.1a}$$

and

$$\bar{\bar{v}}^{\mathbf{a}}_{kl}(x,\omega) = v^e_{kl}(x) \Rightarrow v^{\mathbf{a}}_{kl}(x,t) = v^e_{kl}(x)\delta(t) \tag{3.1b}$$

where $\delta(t)$ is the Dirac delta function. These are Cases 1.1 and 2.1 in Tables 1 and 2 and indicate two radically different results.

If one looks next at Cases 1.2 and 2.2, where the only alteration is that now $\sigma_{kl}(x,t) = g(t)\sigma^*_{kl}(x)$, then one obtains again the above results eqns (3.1a) and (3.1b). Cases 1.3 and 2.3 for $\sigma_{kl}(x,t)$ which are nonseparable functions, yield

$$v_{kl}(x,t) = f_{kl}(x,t) \quad \text{and} \quad v^{\mathbf{a}}_{kl}(x,t) = F_{kl}(x,t) \tag{3.2}$$

where $f_{kl} \neq F_{kl}$.

The similar Cases 1.13–1.15 and 2.13–2.15 for anisotropic viscoelastic compliances with distinct time functions, respond with strikingly different v_{kl} and $v^{\mathbf{a}}_{kl}$ because these time functions no longer divide out in either ratio definition. All six cases, which are for isothermal uniaxial stresses, show that both Poisson ratios are time dependent and only in Cases 1.13 and 2.13 with constant stresses are load history independent PR.

Cases 1.4, 1.8, 1.12, 1.16, 1.20, 1.24, 2.4, 2.8, 2.12, 2.16, 2.20 and 2.24 all include thermal expansions and indicate that under all conditions ν_{kl} and ν^{a}_{kl} are always time, stress and thermal expansion dependent.

Table 3 lists the properties of elastic Poisson ratios. The time dependence is not indicated because elastic material properties are not functions of time and the elastic ν^{e}_{kl} become implicitly time dependent only through their stress functionality. It is readily seen that only isothermal uniaxial stresses lead to constant elastic νs. In all other cases they are stress and thermal expansion dependent, when the latter are present.

Tables 1, 2 and 3 summarize the analytical results for viscoelastic isotropic and anisotropic physical and pseudo Poisson ratios as well as for elastic ones. The FT inversions of eqns (2.8)–(2.10) and (A4)–(A6) yielding $\nu_{kl}(x, t)$ represent formidable tasks and cannot be accomplished and analytically even for the simplest model characterizations. However, they can be carried out numerically with the help of fast FT (FFT) algorithms, particularly if the real time interval of interest is relatively short. In the absence of any inversions, several important observations on the true physical viscoelastic ν_{kl} are immediately evident from the preceding analysis, such as:

(A) Because of their nonlinear relations to strain tensors, neither physically defined isotropic nor anisotropic viscoelastic Poisson ratios can be formulated in terms of the elastic/viscoelastic integral transform analogy, although moduli remain stress and displacement independent for linear materials.

(B) Both isotropic and anisotropic νs depend on material properties, loading (stress) and thermal expansion histories, but under some special circumstances these history functions factor out of the ν_{kl} expressions.

(C) By virtue of eqns (2.8), anisotropic vs can be time independent if and only if all moduli (compliances, relaxation functions) obey the same time functions in all directions, $\mathscr{T} = 0$ and the stress tensors are constants or are separable into $\sigma_{kl}(x, t) = g(t)\sigma^{*}_{kl}(x)$. This compliance class represents a highly restrictive set of conditions (Hilton 1996).

(D) In multiaxial loading, isotropic vs can be time independent if and only if G and K have identical time functions, $\mathscr{T} = 0$, the temperature $T = T(t)$ only, and the stress tensors are separable à la (C) above. Separation of variables here can only be achieved if $\mathscr{T} = 0$ (Hilton and Russell, 1961).

(E) In uniaxial loading, isotropic νs are time independent if and only if $G/K =$ const. (same time functions) and $\mathscr{T} = 0$.

However, important observations and conclusions can be gleaned from analytic examinations of anisotropic and isotropic Poisson ratios as shown in Figs 2 and 3.

In linear viscoelasticity, of which linear elasticity is a subset, the range of values for which ν can theoretically exist is $-1 \leqslant \nu(t) \leqslant 0.5$, but a number of physical implications must be taken into account. Figure 2 is a composite of influences of Young's (E), shear (G) and bulk (K) moduli, and Poisson ratios (ν). In the ν-E/G domain, the region bounded by $2 \leqslant E/G \leqslant 3$ and $0 \leqslant \nu \leqslant 0.5$ normal strains are of opposite signs for 1-D stresses, i.e. tensile $\varepsilon_{11} > 0$ and compressive $\varepsilon_{22} < 0$, etc. or vice versa, which is the usual material behavior, although there are some rubbery materials that are capable of producing equal sign strains in perpendicular directions. The region $E/G \leqslant 2$ where $K < G$ is a physical stranger to most real materials.

Similar, but more complicated relations for $\overline{\overline{\nu}}_{kl}$ in terms of the total set of $\overline{\overline{E}}_{klmn}$ can be derived for general anisotropic viscoelastic states as defined by eqns (2.1), which now additionally involve the shear moduli. In any case, viscoelastic Poisson ratios depend on moduli function fractions as seen from eqns (2.8)–(2.10). If all moduli $E_{klmn}(x, t)$ and relaxation functions $\phi_{klmn}(x, t)$ have identical time functions, then all $\nu_{kl}(x, t)$ are time independent. This is due to the time function transforms in each complex moduli simply factoring and then dividing out of the expressions for $\overline{\overline{\nu}}$ in eqns (2.8)–(2.10). It has been noted by Hilton (1964a, 1996) that such constant viscoelastic Poisson ratio values are physically true only for imcompressible materials when $\nu(t) = 0.5$. Under all other circumstances constant ν values represent extremely restrictive conditions for real materials.

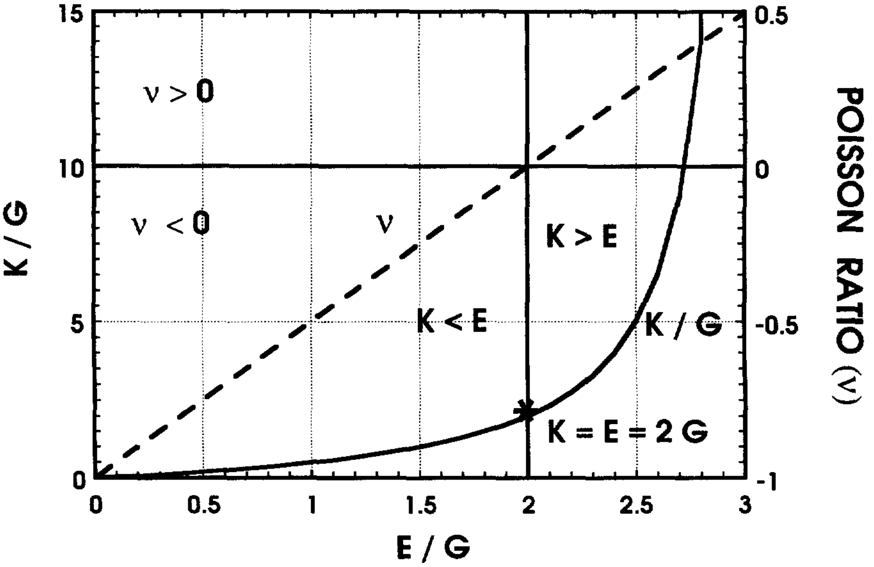

Fig. 2. Interdependence of *E*, *G*, *K* and ν.

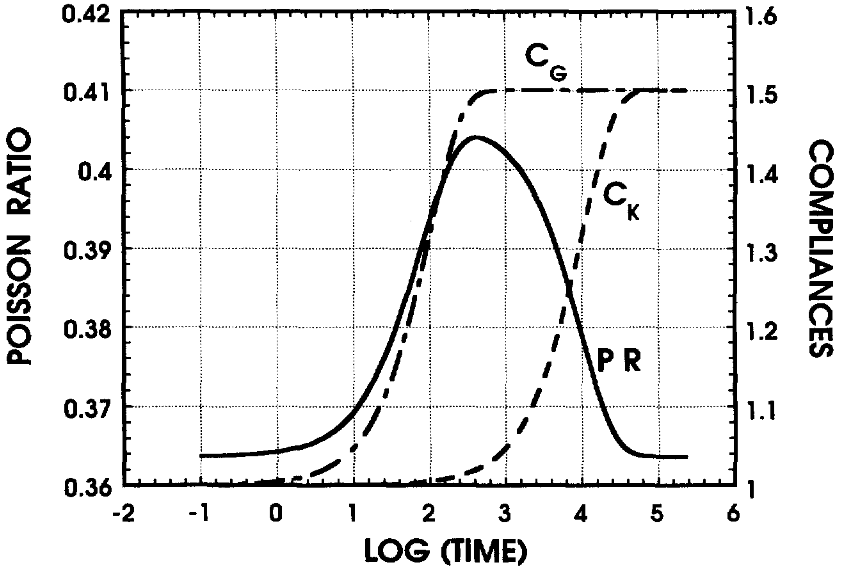

Fig. 3. Poisson ratios and compliances.

Under uniaxial loading ($\sigma_{11} \neq 0$), the FT of isotropic νs in terms of the two moduli is

$$\bar{\nu}(x,\omega) = \int_{-\infty}^{\infty} \left\{ \frac{\displaystyle\int_{-\infty}^{\infty} \left[\frac{1}{2\bar{\bar{G}}(x,\omega)} - \frac{1}{\bar{\bar{K}}(x,\omega)} \right] \bar{\bar{\sigma}}_{11}(x,\omega)\exp(i\omega t)\,\mathrm{d}\omega}{\displaystyle\int_{-\infty}^{\infty} \left[\frac{1}{\bar{\bar{G}}(x,\omega)} + \frac{1}{\bar{\bar{K}}(x,\omega)} \right] \bar{\bar{\sigma}}_{11}(x,\omega)\exp(i\omega t)\,\mathrm{d}\omega} \right\} \exp(-i\omega t)\,\mathrm{d}t \tag{3.3a}$$

or

$$\bar{\bar{v}}(x,\omega)=\int_{-\infty}^{\infty}\left\{\underbrace{\frac{\displaystyle\int_0^t[0.05C_G(x,t,t')-C_K(x,t,t')]\sigma_{11}(x,t')\,\mathrm{d}t'}{\displaystyle\int_0^t[C_G(x,t,t')+C_K(x,t,t')]\sigma_{11}(x,t')\,\mathrm{d}t'}}_{=v(x,t)}\right\}\exp(-i\omega t)\,\mathrm{d}t \tag{3.3b}$$

where the shear and bulk compliances are $\bar{\bar{C}}_G=1/\bar{\bar{G}}$ and $\bar{\bar{C}}_{K^-}=1/\bar{\bar{K}}$. Isotropic compliances can be expressed in the same form eqn (2.12) as anisotropic ones, yielding

$$C_G(x,t)=C_\infty^G(x)\delta(t)-\sum_{p=1}^{N^G}C_p^G(x)\exp[-t/\tau_p^G(x)] \tag{3.4}$$

and

$$C_K(x,t)=C_\infty^K(x)\delta(t)-\sum_{p=1}^{N^K}C_p^K(x)\exp[-t/\tau_p^K(x)] \tag{3.5}$$

Then as an illustrative example, consider the simplest model given by eqns (2.14) with $\tau_p^K=a\tau_p^G$ and $N^G=N^K=N$, and for $\sigma_{11}=\text{const}$ it yields before the FT is applied in eqn (3.3b)

$$v(x,t)=\frac{0.5C_\infty^G(x)-0.5\sum_{p=1}^{N}C_p^G\tau_p[\exp(-t/\tau_p)-1]-C_\infty^K(x)+\sum_{p=1}^{N}(C_p^K\tau_p/a)[\exp(-at/\tau_p)-1]}{C_\infty^G(x)-\sum_{p=1}^{N}C_p^G\tau_p[\exp(-t/\tau_p)-1]+C_\infty^K(x)-\sum_{p=1}^{N}(C_p^K\tau_p/a)[\exp(-at/\tau_p)-1]} \tag{3.6}$$

If $G(x,t)$ are expressed in terms of identical time functions, say $G(x,t)=G_0(x)f(t)$ or $G(x,t)=G_0f(x,t)$ and $K(x,t)=K_0(x)f(t)$ or $K(x,t)=K_0f(x,t)$, then after the inner integrations the inverses of the product functions $\bar{\bar{f}}\bar{\bar{\sigma}}_{11}$ factor and v becomes

$$v(x,t)=v^e(x)=\frac{0.5-G_0(x)/K_0(x)}{1+G_0(x)/K_0(x)} \tag{3.7}$$

where v^e is the time independent elastic (instantaneous) PR. Equation (3.6) can also be interpreted by setting $a=1$ and $C_K=AC_G$, which again yields the same time independent PR of eqn (3.7). Similarly, the anisotropic vs of the previous section would reduce to their elastic counterparts, with $v_{kl}(x,t)=v_{kl}^e(x)$ for $k\neq l$, when the moduli have the same time functions in all directions, such that $E_{klmn}(x,t)=E_{klmn}^0(x)f(t)$ or $E_{klmn}(x,t)=E_{klmn}^0f(x,t)$.

Figure 3 is a plot of isotropic C_G, C_K and v vs log time. All curves represent a real material and are modeled with $N=29$. The C_G and C_K are normalized w.r.t. their own elastic instantaneous (maximum) values. Numerical values are $C_\infty^G/C_0^G=C_\infty^K/C_0^K=1.5$, $C_0^K/C_0^G=0.01$ and $a=0.01$. It can be readily seen that v is time independent only in those t regions where C_0^K/C_0^G or C_∞^K/C_∞^G are constants. This condition indicates that both changes in shape and in volume must proceed either in an elastic manner or in a fully relaxed fashion for $v(t)$ to remain constant. Since in the present illustrative example the ratios G_0/K_0 and G_∞/K_∞ are equal, it follows that $v_0=v_\infty$ and in the present example they are equal to 0.364.

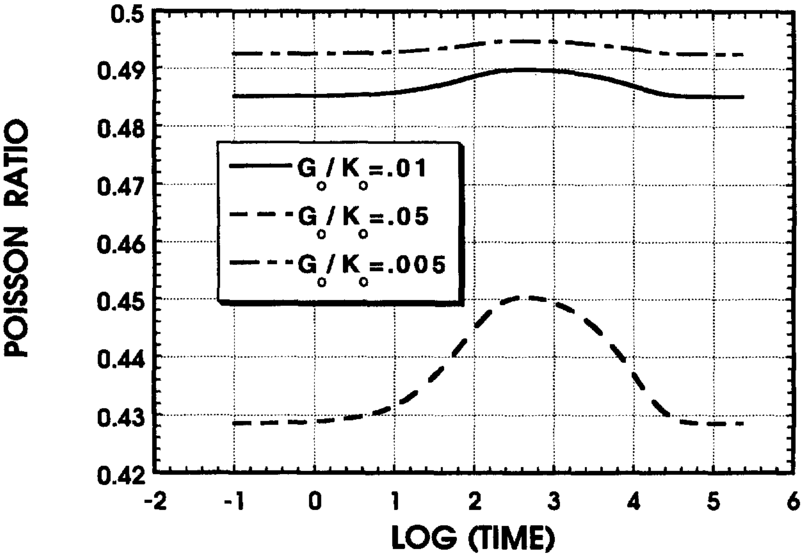

Fig. 4. Effects of initial moduli on Poisson ratios.

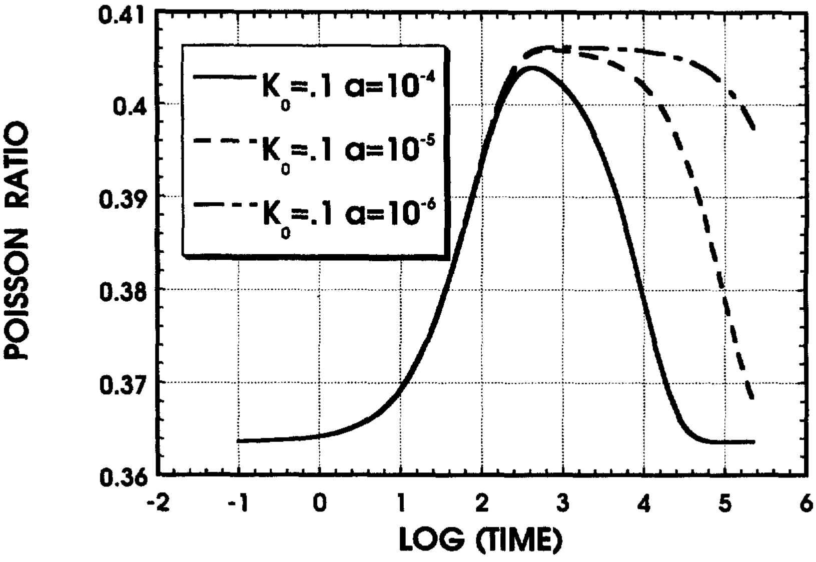

Fig. 5. Effects of time shifts on Poisson ratios.

Figures 4 and 5, respectively, depict the effects of G_0/K_0 and of a on PR. It is seen the larger the K_0 compared to G_0 the less time dependent v becomes and approaches the incompressible value of 1/2. The influence of decreasing values of a increase the relaxation times τ_p^K compared to τ_p^G and to widen the number of decades over which PR are time dependent.

The relative influences of Gs and Ks on isotropic PR of eqns (3.6) and in Figs 4 and 5 are identical to that of C_{ll11} and C_{kk11} on anisotropic v_{kl} of eqns (2.6) for $\mathscr{A}\mathscr{T} = 0$ and uniaxial stress fields ($\sigma_{11} \neq 0$).

One must now ask what kind of real materials admit constant Poisson ratios to characterize their viscoelastic behavior. One obvious answer is all incompressible isotropic materials do, where $v(t) = 0.5$. For isotropic materials, eqns (3.3) and (3.4) require that G and K obey the same time functions so that

$$\frac{G(x,t)}{K(x,t)}=\frac{G_0(x)}{K_0(x)} \tag{3.8}$$

It is an observable fact that eqns (3.4) and (3.5) are not satisfied by real viscoelastic materials, except for the incompressible ones or where $\nu \simeq 0.4999^+$ and $K_0 \gg G_0$. For most materials K_0 is only $> G_0$ but additionally real viscoelastic materials, such as high temperature metals, polymers, composites, etc., exhibit creep and relaxation rates for volume changes which are much slower than those contributing to shape changes, indicating $\tau_p^K > \tau_p^G$. Consequently, on a time plot $C_K(t)$ curves are always shifted in the direction of larger t when compared to $C_G(t)$. Conversely, since $\omega \sim 1/t, \overline{C}_K(\omega)$ lags behind $\overline{C}_G(\omega)$ on a frequency ω plot.

Similarly, in the anisotropic case eqns (2.8) to (2.10) again mandate that in order to achieve constant $\nu_{kl}(t)$ values all pertinent $E_{klmn}(t)$ must be in either their elastic or fully relaxed ranges as $t \to 0$ or $t \to \infty$.

Physically, constant viscoelastic ν_{kl} conditions over the entire time space can be approached if real materials such as, for instance, composites have fibers which either are elastic or have extremely slow creep rates with relaxation times orders of magnitude larger compared to the viscoelastic polymer matrix. The values of the E_{ijkl} in different orientations are then predominately dictated by fiber stiffnesses and directional arrangements, while the polymer matrix remains isotropically viscoelastic with identical constitutive relations in all directions.

The analysis indicates that isotropic viscoelastic materials require two moduli obeying different time functions, while anisotropic materials need as few as nine and as many as 21 time dependent moduli for complete property specification. Additionally and independently, one thermal expansion function is called for in isotropic viscoelasticity and one to six are needed for anisotropic modeling.

Ultimately, it must be remembered that Poisson ratios do not contribute anything additional to material characterization that cannot be achieved totally by moduli, compliances, or creep and relaxation functions. The utility of the PR because of their nonlinear behavior is, therefore, severely limited. Consequently, moduli or compliances, or creep or relaxation functions should be the vehicle of choice for describing viscoelastic responses.

Isotropic and anisotropic viscoelastic analyses in the present paper demonstrate the close relationship between viscoelastic moduli and Poisson ratios, particularly in regard to their time and stress dependences. No experimental data for non-metallic materials seems to exist describing complete viscoelastic multiaxial experimental strain or displacement measurements necessary for proper Poisson ratio determinations. The works of Beckwith (1980, 1984), Bogetti and Gillespie (1991), Crossman *et al.* (1978), Flaggs and Crossman (1981), Halpin and Pagano (1968), Lin and Hwang (1989), Tuttle and Brinson (1985a, b, 1986) and Yang *et al.* (1983), among other, are typical examples of experimental investigations reporting on anisotropic moduli from which, because of inherent 2-D and 3-D experimental complexities it is difficult to deduce the total anisotropic modulus and Poisson ratio picture. Additionally, an extensive collection of isotropic viscoelastic modulus data has been presented and analyzed by Nashif *et al.* (1985).

Levitsky and Shaffer's (1974) and Shaffer and Levitsky's (1974) interpretations of responses for elastic thermo-setting materials use Poisson ratios which are not defined according to the present multiaxial analysis. Subsequently, Levitsky and Shaffer (1975) employed a single Maxwell Viscoelastic model with constant bulk modulus and a PR defined by

$$\nu(t)=\frac{0.5-G(t)/K_0}{1+G(t)/K_0} \tag{3.9}$$

which is inapplicable for their 2-D time dependent stress states. Finally, Kim and White

(1996, 1997) assumed a constant Poisson ratio $\nu_0(t)$ during AS4/3501-6 composite cure and then calculated a relaxation modulus $E(t)$ based on the relation

$$E(t) = 2(1+\nu_0)G(t) \tag{3.10}$$

which essentially begs the equation regarding appropriate viscoelastic values of ν and renders the calculation of $E(t)$ from experimentally determined $G(t)$ or vice-versa highly questionable for their multiaxial time dependent stress fields. It is unfortunate that these five publications chose an involvement with ill defined Poisson ratios, because their analyses and interpretations of experimental data could be carried out in terms of moduli without ever appealing to any PR concepts. As a matter of fact, viscoelastic material characterization is best defined in terms of moduli or compliances, or creep or relaxation functions, which for linear materials do not bear the nonlinear and load-time history burden of Poisson ratios. Hwang (1990) has reported viscoelastic experimental results for IM7/851-7 graphite/epoxy composites indicating that in the fiber dominated direction the compliances C_{1111} and Poisson's ratio ν_{12} are time independent, while the compliances C_{2222} and C_{2323} exhibit distinct time dependent responses.

4. CONCLUSIONS

The isotropic and anisotropic viscoelastic analyses demonstrate the close relationship between viscoelastic moduli and Poisson ratios, particularly in regard to the latter's time dependence and nonlinear functionality w.r.t. stress fields, thermal expansions and time histories. Uniaxially determined Poisson ratios are not exportable to other uniaxial conditions with different loading time sequences nor are they applicable to multi-axial stress fields. Poisson ratios add little to viscoelastic isotropic and anisotropic material characterization that is not already encompassed by moduli, compliances, and creep and relaxation functions. Consequently, any one of the latter should be the characterization model of choice, particularly because of the inherent nonlinear nature of Poisson ratios. No experimental data for non-metallic materials seems to exist describing multiaxial experimental strain or displacement measurements necessary for proper Poisson ratio determinations. It is, therefore, recommended that multiaxial experiments and data collection be undertaken on viscoelastic materials in order to correctly catalogue isotropic and anisotropic deterministic and stochastic material property parameters.

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APPENDIX A—ALTERNATE POISSON RATIO MODELING

It is possible to postulate an alternate model for a pseudo Poisson ratio, ν^{a}_{kl}, based on a revised eqn (2.7) as (see for instance Zinoviev and Ermakov (1994) and others).

$$\bar{\bar{\nu}}^{a}_{kl}(x,\omega) = -\frac{\bar{\bar{\varepsilon}}_{ll}(x,\omega)}{\bar{\bar{\varepsilon}}_{kk}(x,\omega)} \quad \text{and} \quad \nu^{a}_{kl} \neq \nu^{a}_{lk} \quad \text{for } k \neq l \tag{A1}$$

While this $\bar{\bar{\nu}}^{a}_{kl}$ has no relation to the proper physical $\bar{\bar{\nu}}_{kl}$ defined by the FT of eqn (2.6), it presents some computational advantages and a certain symmetry in relation to corresponding elastic $\bar{\bar{\nu}}^{e}_{kl}$. As a first illustrative example, consider an isotropic elastic medium, where

$$\nu^{e}(x) = \nu_0(x) = \frac{0.5 - G_0(x)/K_0(x)}{1 + G_0(x)/K_0(x)} \tag{A2}$$

The usual protocol for applying the elastic–viscoelastic analogy to ν, and in this instance without regard for the definition (2.6), is to take the FT of eqn (A2) and substitute complex viscoelastic moduli or compliances for elastic ones, such that

$$\bar{\bar{\nu}}^{a}_{kl}(x,\omega) = \frac{0.5 - \bar{\bar{G}}(x,\omega)/\bar{\bar{K}}(x,\omega)}{1 + \bar{\bar{G}}(x,\omega)/\bar{\bar{K}}(x,\omega)} \quad k \neq l \tag{A3}$$

and

$$\bar{\bar{\nu}}^{a}_{12}(x,\omega) = \bar{\bar{\nu}}^{a}_{13}(x,\omega) \quad \text{and} \quad \bar{\bar{\nu}}^{a}_{23}(x,\omega) = 1$$

with $\bar{\bar{\nu}}^{a}_{kl} \neq \bar{\bar{\nu}}_{kl}$ of eqns (2.9) and (2.10), unless both PR are time dependent. The FT only exists only if the previously discussed caveats regarding $\mathscr{T} = 0$ or thermorheologically simple materials with equal ξ_{klmn} are in force. Therefore, the definitions and results of this Appendix as well as the conclusions and comparisons to ν_{kl} in the Discussion Section apply only if the FT of eqns (A1) exits. The expressions (A3) can also be obtained directly from the definitions (A1) and (2.11) for any uniaxial stress $\sigma_{11}(x,t)$. When the expression for $\bar{\bar{\nu}}^{a}_{kl}$ is inverted, it yields

$$v^{\mathbf{a}}_{kl}(x,t) = \int_{-\infty}^{\infty} \bar{\bar{v}}^{\mathbf{a}}_{kl}(x,\omega)\exp(i\omega t)\,\mathrm{d}\omega \tag{A4}$$

and bears no resemblance to the proper physical definition of v_{kl} from eqns (2.9) and (2.10) since these alternate ones are devoid of any loading and thermal expansion time histories.

Similarly, for the isothermal uniaxial loading in the x_1 direction of eqns (2.2b)–(2.4b), pseudo Poisson ratios can be defined for an orthotropic material as

$$\bar{\bar{v}}^{\mathbf{a}}_{12}(x,\omega) = -\frac{\bar{\bar{C}}_{12}}{\bar{\bar{C}}_{11}} \quad \bar{\bar{v}}^{\mathbf{a}}_{13}(x,\omega) = -\frac{\bar{\bar{C}}_{13}}{\bar{\bar{C}}_{11}} \quad \bar{\bar{v}}^{\mathbf{a}}_{23}(x,\omega) = -\frac{\bar{\bar{C}}_{13}}{\bar{\bar{C}}_{12}} \tag{A5}$$

Unfortunately, the simplicity of eqns (A5) is deceiving as its stress and thermal expansion free definition is caused by the uniaxial stress state. For multiaxial nonisothermal conditions, eqns (2.1b) substituted into (A1) yield the general expression for the pseudo Poisson ratio as

$$\bar{\bar{v}}^{\mathbf{a}}_{kl}(x,\omega) = -\frac{\bar{\bar{C}}_{\underline{ll}mn}(x,\omega)\bar{\bar{\sigma}}_{mn}(x,\omega) + \bar{\bar{q}}_{\underline{ll}}(x,\omega)\mathscr{A}\bar{\bar{\mathscr{T}}}(x,\omega)}{\bar{\bar{C}}_{\underline{kk}op}(x,\omega)\bar{\bar{\sigma}}_{op}(x,\omega) + \bar{\bar{q}}_{\underline{kk}}(x,\omega)\mathscr{A}\bar{\bar{\mathscr{T}}}(x,\omega)} \quad k \neq l \tag{A6}$$

indicating that if more than one stress tensor component is present, even for $\mathscr{T} = 0$, $v^{\mathbf{a}}_{kl}$ is not stress history independent. The FT inversion of eqns (A6)—(A4) is considerably more involved than that of eqns (2.8) for v_{kl}. A detailed analysis of the behavior of eqns (A6) are given in Table 2 and in the Discussion Section.